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Numerical verification method of solutions for nonlinear elliptic and evolutional problems

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Abstract

We consider the methods for guaranteed computations of solutions for nonlinear parabolic initial-boundary value problems. First, in order to make the basic principle clear, we briefly introduce the numerical verification methods of solutions for elliptic problems which we have developed up to now. Next, under some fundamental procedures of verification for parabolic problems based on the fixed point theorem with Newton's method, we describe a summary of our methods including additional new technique which could yield some improvements. The main contents of the paper consist of the guaranteed a posteriori estimates for the linearized inverse operators of parabolic type. In order to confirm the effectiveness of our methods, we give some numerical examples for the guaranteed bounds of iverse operators as well as give some prototype results for numerical verification of solutions of nonlinear parabolic problems. Moreover, we will mention an extention of the present technique to the verification of time-periodic solutions.

Keywords: Numerical Verification Methods, Elliptic boundary value problem, Parabolic initial-boundary value problem, Fixed point theorem, Newton's method

1 Introduction

Since, we will mainly concern the arguments related to the Newton-type verifiaction methods, our main task consists of the way how to estimate the norm of a linearized inverse operator for the original nonlinear problem. Therefore, in this paper, we focus on the constructive a priori estimates of solution for linear parabolic problems, which we sometimes call *'a posteriori estimates'* of the inverse operator.

In [15, 17], solving parabolic problems with guaranteed error bounds were considered based on the sequential iteration method with Schauder's fixed point theorem. In these cases, due to the verification principle, concerned operator is implicitly supposed to be retractive in a neighborhood of the solution. As well known, e.g.[31], for a linear parabolic operator $\mathscr{L}_t u \equiv u_t - \nu\Delta u + (b\cdot\nabla)u + cu$, the theoretical norm estimates of the inverse operator $\mathscr{L}_t^{-1}$ is generally exponentially dependent on the time interval. Therefore, it should be not efficient to use such a norm for the actual implementation of our verification procedures based on the Newton method. In [12, 13], some verification procedures are presented by using the theoretical estimates similar to that in [31], but the verification cost seems to be high provided that the corresponding elliptic part, .i.e., $-\nu\Delta u + (b\cdot\nabla)u + cu$, is not coercive. On the other hand, in the results by [1, 32] on the numerical verifications of time-periodic solutions in one space dimension, they use the spectral method with explicit eigenvalues and corresponding eigen-functions for the linear part of concerned nonlinear equation. Their methods are fairly different from the present approach based on the finite element methods.

In the present paper, after some consideration of the theoretical estimates, we will introduce two kinds of a posteriori techniques. One of them, proposed in [23], uses a constructive error estimates for spatial semidiscrete approximation by the finite element method to the simple heat equation as well as the inverse operator estimates for a linear ordinary differential system obtained by the semidiscretization. Another method uses a full-discrete numerical scheme which is based on an interpolation in time by using the fundamental solution for spatial discretization. In this technique, the constructive a priori error estimates for a full discretization of solutions to the heat equation play an essential role, which is considered as the same situation in the elliptic case([19, 22, 29] etc.). In both techniques, combining these estimates with an argument using the discretized inverse operator and the condition of contraction in the Newton-type formulation, the a posteriori estimates of the norm for the infinite-dimensional operators are presented.

In order to clarify the basic and essential concepts of our numerical verification techniques, first we give a brief summary of our method for the elliptic boundary value problems in Section 2. Next, in Section 3, after describing a verification principle for parabolic problems using Newton-type formulation, according to our results [8, 23, 24], we introduce three kinds of method for the estimation of inverse operator $\mathscr{L}_t^{-1}$ and compare them by some numerical examples. Also we show some examples of the verification for solutions of a prototype nonlinear problem. Additionally, we mention a possible refinement on the estimation for the inverse operators. In Section 4, we will consider a basic formulation of the numerical enclosing time-periodic solutions for parabolic problems with known and unknown periods. We summarize the paper in Section 5.

2 Elliptic problems

In this section, we briefly describe the basic principles for the numerical verification of solutions to the following elliptic problems, see [14, 16, 18, 19] etc. for details,

$$\begin{cases} -\Delta u &= f(x,u,\nabla u), \quad x\in\Omega, \\ u &= 0, \quad x\in\partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain in R^d $(1\le d\le 3)$, f is a nonlinear map. We use the homogeneous Sobolev space $H_0^1(\Omega)(\equiv H_0^1)$ for the solution of (1). Also some appropriate assumptions are imposed on the map f. In order to treat the problem as the finite procedure on computer, we use a finite dimensional subspace S_h of H_0^1 dependent on a parameter h. Usually, S_h means a finite element subspace on Ω with mesh size h with nodal functions $\{\phi_i\}_{1\le i\le n}$.
Denoting the inner product on $L^2(\Omega)$ by $(\cdot,\cdot)$, we define the H_0^1-projection: $P_h\phi\in S_h$ for $\phi\in H_0^1$, by

$$(\nabla\phi-\nabla(P_h\phi),\nabla v_h)=0, \quad \forall v_h\in S_h. \tag{2}$$

If $\Delta\phi\in L^2(\Omega)$, then the following a priori error estimates plays an essential role to bridge between the infinite and finite dimensional, i.e., continuous and discrete, problems.

$$\|(I-P_h)\phi\|_{H_0^1}\le C(h)\|\Delta\phi\|_{L^2}. \tag{3}$$

Here, I stands for the identity on H_0^1 and $C(h)$ means a positive constant which can be numerically determined such that $C(h)\to 0$ as $h\to 0$. For example it can be taken as $C(h)=h/\pi$ and $h/(2\pi)$ for bilinear and biquadratic element, respectively, for the rectangular mesh on the square domain [6], and $C(h)=0.493h$ for the linear and uniform triangular mesh of the convex polygonal domain [5]. Even for the nonconvex domain, we can also numerically give such a constant, e.g., [30, 9, 10, 28].

For each $\psi\in L^2(\Omega)$, we denote a solution $\phi\in H_0^1$ of the Poisson equation : $-\Delta\phi=\psi$ with homogeneous boundary condition by $\phi\equiv(-\Delta)^{-1}\psi$. Then, under some appropriate conditions on f, (1) is rewritten as the fixed point equation of the form $u=F(u)$ with a compact map $F\equiv(-\Delta)^{-1}f$ on H_0^1.
The following decomposion of the fixed point equation gives an essential principle which enables us to treat the problem by finite computational procedures.

$$\begin{cases} P_h u &= P_h F(u), \\ (I-P_h)u &= (I-P_h)F(u). \end{cases} \tag{4}$$

Here, the first and second parts can be considered as the equations in S_h and in the orthogonal complement $S_h^{\perp}$ of H_0^1, respectively.

Sequential iterative method A set $U \subset H_0^1$ which possibly includes a solution of (4) is called *a candidate set* of solutions. Usually, for the sets $U_h \subset S_h$ and $U_{\perp} \subset S_h^{\perp}$, the candidate set U is taken as $U = U_h \oplus U_{\perp}$. Then, a verification condition based on Schauder's fixed point theorem is given by

$$\left\{ \begin{array}{lcl} P_h F(U) & \subset & U_h, \\ (I - P_h) F(U) & \subset & U_{\perp}. \end{array} \right. \tag{5}$$

The set U_h is taken to be a set of linear combinations of basis functions in S_h with interval coefficients, while $U_{\perp}$ a ball in $S_h^{\perp}$ with radius $\alpha \geq 0$.
Note that, it can be easily seen that $P_h F(U)$ is directly computed or enclosed for given U_h and $U_{\perp}$ by solving a linear system of equations with interval right-hand side using some interval arithmetic approaches. On the other hand, $(I - P_h)F(U)$ can be evaluated as a positive real number by the use of constructive a priori error estimates (3) of the form

$$||(I - P_h)F(U)||_{H_0^1} \leq C(h) \sup_{u \in U} ||f(u)||_{L^2}. \tag{6}$$

Thus, the former condition in (5) is validated as the inclusion relations of corresponding coefficient intervals, and the latter part can be confirmed by comparing two nonnegative real numbers which correspond to the radii of balls. In order to find the candidate set U in the actual computations, some iterative methods are effectively utilized([14]).

Finite dimensional Newton's method Note that the verifiaction condition (5) is not applicable except that the concerned operator F is retractive around the fixed point. Therefore, in order to overcome this difficulty, we need some Newton-type method for (4). Thus, we define the nonlinear operator N_h with an approximate solution $\widehat{u}_h$ by

$$N_h(u) := P_h u - [P_h - P_h A'(\widehat{u}_h)]_h^{-1} (P_h u - P_h F(u)),$$

where $A'(\widehat{u}_h) \equiv (-\Delta)^{-1} f'(\widehat{u}_h)$ and $'$ means the Fréchet derivative of f at $\widehat{u}_h$. Here, $[P_h - P_h A'(\widehat{u}_h)]_h^{-1}$ denotes the inverse on S_h of the restriction operator $(P_h - P_h A'(\widehat{u}_h))|_{S_h}$. The existence of such a finite dimensional inverse operator can be validated by the usual invertibility of the corresponding matrix. And we set

$$T(u) \quad := \quad N_h(u) + (I - P_h)F(u).$$

Then T is considered as the Newton-type operator for the former part of (4) but the simple iterative operator for the latter part. It can be seen that $u = T(u)$ is equivalent to $u = F(u)$, and the verification condition is presented similar as before.

Infinite dimensional Newton's mtheod By applying the verification principle to the linearized equation for the original problem (1), we can also realize an infinite dimensional Newton-type mtheod. We now assume that the linearlized equation at $\widehat{u}_h$ is written as

$$\left\{ \begin{array}{rl} \mathscr{L}u := -\Delta u + b \cdot \nabla u + cu = \psi, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{array} \right. \tag{7}$$

Here, we assume that $b \in W_1^\infty(\Omega)^d$, $c \in L^\infty(\Omega)$, $\psi \in L^2(\Omega)$. Also define the matrices $G = (G_{i,j})$ and $D = (D_{i,j})$ by :

$$\begin{aligned} G_{i,j} &= (\nabla\phi_j, \nabla\phi_i) + (b\cdot\nabla\phi_j, \phi_i) + (c\phi_j, \phi_i), \\ D_{i,j} &= (\nabla\phi_j, \nabla\phi_i), \qquad \text{for} \quad 1 \le i, j \le n. \end{aligned}$$

Let define $\rho := \|D^{\frac{T}{2}} G^{-1} D^{\frac{1}{2}}\|_E$, which implies the Euclidean norm of the matrix. Then ρ corresponds to an approximate operator norm for $\mathscr{L}^{-1}$ in H_0^1 sense. Setting the constants as $C_{\mathrm{div}\,b} := \|\mathrm{div}\,b\|_{L^\infty(\Omega)}$, $C_b := \left(\sum_{i=1}^d \|b_i\|^2_{L^\infty(\Omega)}\right)^{1/2}$, $C_c := \|c\|_{L^\infty(\Omega)}$, and let $\tilde{C}_1 := C_p C_{\mathrm{div}\,b} + C_b$, $\tilde{C}_2 := C_p C_c$, $\tilde{C}_3 := C_b + C_p C_c$, $\tilde{C}_4 := C_b + C(h) C_c$, where C_p is a Poincaré constant on Ω. Then, we have the following invertibility condition for $\mathscr{L}$ in (7).

Theorem 2.1 ([19]). *If*

$$\kappa \equiv C(h)\big(\rho\tilde{C}_3(\tilde{C}_1 + \tilde{C}_2)C(h) + \tilde{C}_4\big) < 1, \tag{8}$$

then the operator $\mathscr{L}$ in (7) is invertible. Here, $C(h)$ is the same constant in (3).

By using this result we derive a verificaion condition for the solution of the problem (1) to apply the infinite dimensional Newton-type method.
On the other kind of verification methods for elliptic problems, refer [3, 25, 26, 28] and so on.

3 Parabolic initial-boundary value problems

We consider the following parabolic initial-boundary value problems:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu\Delta u = f(x,t,u,\nabla u) & \text{in } \Omega \times J, \quad (9a) \\ u(x,t) = 0 & \text{on } \partial\Omega \times J, \quad (9b) \\ u(x,0) = 0 & \text{in } \Omega, \quad (9c) \end{cases}$$

where $x \in \Omega \subset R^d$: a bounded convex domain, $t \in J := (0,T) \subset R$: a bounded interval for a fixed T, and $\nu \in R$: a positive constant. We assume that f is a continuous map from $L^2(J;H_0^1(\Omega))$ into $L^2(J;L^2(\Omega))$, and, for each bounded subset U in $L^2(J;H_0^1(\Omega))$, the image of U by f is also bounded in $L^2(J;L^2(\Omega))$.

In order for the verified computation of solutions for (9), in [15, 17], some sequential iterative methods similar to that in the previous section are used. And not yet considered for the finite dimensional Newton-type method, as in the second paragraph of the section 2, up to now. From the fact that the invertibility of the linearlized operator for parabolic equation is always valid, some other Newton type method is studied in [12, 13]. However, the numerical examples in those works are prototype problems and it is not clear whether they can be applied to more realistic problems. In the present section, we describe three kinds of method, one a priori and two a posteriori, to compare the efficiency by showing some numerical data.

We now define the space by

$$V^1(J;L^2(\Omega)) := \left\{u \in L^2(J;L^2(\Omega))\ ;\ \frac{\partial u}{\partial t} \in L^2(J;L^2(\Omega)),\ u(\cdot,0) = 0 \text{ in } L^2(\Omega)\right\}$$

with inner product $(u,v)_{V^1(J;L^2(\Omega))} := \left(\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}\right)_{L^2(J;L^2(\Omega))}$. By using an appropriate approximate solution $u_h^k \in V^1(J;L^2(\Omega)) \cap L^2(J;H_0^1(\Omega) \cap H^2(\Omega))$ and setting $u \equiv w + u_h^k$, the original problem (9) can be rewritten in the following residual form

$$
\begin{cases}
\mathscr{L}_t w = g(w) & \text{in } \Omega \times J, & (10a)\\
w(x,t) = 0 & \text{on } \partial\Omega \times J, & (10b)\\
w(x,0) = 0 & \text{in } \Omega, & (10c)
\end{cases}
$$

where $\mathscr{L}_t := \frac{\partial}{\partial t} - \nu\Delta - f'(u_h^k)$. Here, $f'(u_h^k)$ stands for a Fréchet derivative of f at u_h^k. And $g(w) \equiv f(x,t,w+u_h^k,\nabla(w+u_h^k)) - \frac{\partial u_h^k}{\partial t} + \nu\Delta u_h^k - f'(u_h^k)w$.
In order to consider the existence of a solution w of (10), for any $\alpha > 0$, we define the candidate set by

$$W_\alpha := \{w \in L^2(J;H_0^1(\Omega));\ \|w\|_{L^2(J;H_0^1(\Omega))} \le \alpha\}. \tag{11}$$

If we find a constant $C_{\mathscr{L}_t^{-1}}$ satisfying

$$\|\mathscr{L}_t^{-1}\|_{\mathscr{L}(L^2(J;L^2(\Omega)),L^2(J;H_0^1(\Omega)))} \le C_{\mathscr{L}_t^{-1}}, \tag{12}$$

then, we have

$$\|\mathscr{L}_t^{-1}g(W_\alpha)\|_{L^2(J;H_0^1(\Omega))} \le C_{\mathscr{L}_t^{-1}} \sup_{w\in W_\alpha} \|g(w)\|_{L^2(J;L^2(\Omega))}.$$

Therefore, using the compact imbedding from $V^1(J;L^2(\Omega)) \cap L^2(J;H_0^1(\Omega)\cap H^2(\Omega))$ into $L^2(J;H_0^1(\Omega))$, by the Schauder fixed point theorem we obtain the following existential condition of a solution $w \in W_\alpha$ to (10),

$$C_{\mathscr{L}_t^{-1}} \sup_{w\in W_\alpha} \|g(w)\|_{L^2(J;L^2(\Omega))} \le \alpha. \tag{13}$$

This clearly implies a Newton-type verification condition of solutions for the original problem (9).
Note that usually $\mathscr{L}_t$ is written of the form

$$\mathscr{L}_t w \equiv \frac{\partial}{\partial t}w - \nu\Delta w + (b\cdot\nabla)w + cw, \tag{14}$$

where b and c are L^∞ functions on $\Omega \times J$. Hence we now consider the linear problems:

$$
\begin{cases}
\mathscr{L}_t w = q & \text{in } \Omega \times J, & (15a)\\
w(x,t) = 0 & \text{on } \partial\Omega \times J, & (15b)\\
w(x,0) = 0 & \text{in } \Omega, & (15c)
\end{cases}
$$

where the right-hand side q of (15a) means a given function in x and t. Thus, it is important and essential for our purpose to find a constant $C_{\mathscr{L}_t^{-1}}$ satisfying the following a priori estimates of solution w to (15)

$$\|w\|_{L^2(J;H_0^1(\Omega))} \le C_{\mathscr{L}_t^{-1}}\|q\|_{L^2(J;L^2(\Omega))},$$

which also implies that (12) holds for this constant $C_{\mathscr{L}_t^{-1}}$.
In the below, we consider the methods to give such a constant.

3.1 A priori estimates

It is not so difficult to determine the constant $C_{\mathscr{L}_t^{-1}}$ by some theoretical consideration(e.g. [31]), which we call '*a priori estimates*'. However, in general, $C_{\mathscr{L}_t^{-1}}$ obtained by existing a priori methods is exponentially dependent on the time interval J unless that the elliptic part of the operator $\mathscr{L}_t$ is coercive [11]. For example, in case of $b=0$, the following a priori estimate is easily derived [31],

$$\|\mathscr{L}_t^{-1}\|_{\mathscr{L}(L^2(J;L^2(\Omega)),L^2(J;H_0^1(\Omega)))} \leq \exp(\beta T)\frac{C_p}{\nu}, \tag{16}$$

where C_p is a Poincaré constant on Ω and β a nonnegative parameter defined as $\beta \equiv \max\{\sup_{\Omega\times J}(-c),0\}$. Therefore, if the function c takes negative value, then the right-hand side of (16) becomes very large and it leads to an over-estimation of the inverse operator $\mathscr{L}_t^{-1}$. In [13], a weighted norm on the time-dependent Sobolev space is used, but the influence of the exponential dependency on T still remains.

3.2 A posteriori estimates I

Let $S_h(\Omega) \equiv S_h$ be a finite element subspace for the spatial direction with the following approximation properties as in the previous section.

Assumption 1. There exists a constant $C(h)$ such that

$$\begin{aligned}
\|u - P_h u\|_{H_0^1(\Omega)} &\leq C(h)\|\Delta u\|_{L^2(\Omega)}, \quad {}^\forall u \in H_0^1(\Omega) \cap \{\Delta u \in L^2(\Omega)\}, \\
\|u - P_h u\|_{L^2(\Omega)} &\leq C(h)\|u - P_h u\|_{H_0^1(\Omega)}, \quad {}^\forall u \in H_0^1(\Omega),
\end{aligned}$$

where P_h is the H_0^1-projection defined in Section 2.
As well known, this property is valid for many standard finite element subspaces.
Now, we define the several functional spaces :
$V := V^1(J;L^2(\Omega)) \cap L^2(J;H_0^1(\Omega))$ with $(u,v)_V := \left(\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}\right)_{L^2(J;L^2(\Omega))} + \nu(\nabla u, \nabla v)_{L^2(J;L^2(\Omega))}$.
$V^1(J) := \{u \in H^1(J);\, u(0)=0\}$ with $(u,v)_{V^1(J)} := (u',v')_{L^2(J)}$.
Also let $V^1(J;S_h(\Omega)) := \{u_h(x,t) \equiv \sum_{i=1}^{n} a_i(t)\phi_i(x) \mid a_i \in V^1(J),\ 1 \leq i \leq n\}$, and let $P_h^V : V \to V^1(J;S_h(\Omega))$ be a semidiscrete projection defined by, denoting $u(\cdot,t) \equiv u(t) \in H_0^1(\Omega)$,

$$\left(\frac{\partial}{\partial t}(u(t) - P_h^V u(t)), v_h\right)_{L^2(\Omega)} + \nu\left(\nabla(u(t) - P_h^V u(t)), \nabla v_h\right)_{L^2(\Omega)^d} = 0, \quad \forall v_h \in S_h(\Omega), \text{ a.e. } t \in J. \tag{17}$$

For a solution w of (15), setting $w_\perp := (I - P_h^V)w$, the semidiscrete projection (17) leads to the following system of ODEs:

$$\begin{aligned}
&\left(\tfrac{\partial}{\partial t}P_h^V w, v_h\right)_{L^2(\Omega)} + \nu\left(\nabla P_h^V w, \nabla v_h\right)_{L^2(\Omega)^d} + \left((b\cdot\nabla)P_h^V w + cP_h^V w, v_h\right)_{L^2(\Omega)} \\
&\qquad = \left(P_h^0(-(b\cdot\nabla)w_\perp - cw_\perp + g), v_h\right)_{L^2(\Omega)} \\
&\iff \left(L_\phi \frac{d}{dt} + Q_\phi\right)\alpha = L_\phi \beta,
\end{aligned}$$

where P_h^0 means the semidiscrete L^2-projection from $L^2(J;L^2(\Omega))$ into $V^1(J;S_h(\Omega))$, and n-dimensional two vectors α and β are defined by

$$P_h^V w = \sum_{i=1}^{n} \alpha_i(t)\phi_i(x) \quad \text{and} \quad P_h^0(-(b\cdot\nabla)w_\perp - cw_\perp + g) = \sum_{i=1}^{n} \beta_i(t)\phi_i(x),$$

respectively. Here, L_ϕ, Q_ϕ are $n \times n$ matrices such that

$$L_{\phi ij} := (\phi_j, \phi_i)_{L^2(\Omega)},$$

$$Q_{\phi ij} := \nu(\nabla\phi_j, \nabla\phi_i)_{L^2(\Omega)} + ((b\cdot\nabla)\phi_j, \phi_i)_{L^2(\Omega)} + (c\phi_j, \phi_i)_{L^2(\Omega)}.$$

Next, we define the positive constant M_ϕ^{10} which corresponds to the norm of the semidiscrete inverse operator such that

$$\left\| D_\phi^{T/2}\left(L_\phi \frac{d}{dt} + Q_\phi\right)^{-1} L_\phi^{1/2} \right\|_{\mathscr{L}(L^2(J)^n, L^2(J)^n)} \le M_\phi^{10},$$

where, D_ϕ is an $n \times n$ matrix defined as $D_{\phi ij} := (\nabla\phi_j, \nabla\phi_i)_{L^2(\Omega)}$.
Now, by some arguments on the error estimates of semidiscrete projection P_h^V with use of Assumption 1, we have the following theorem.

Theorem 3.1 ([23]). *If*

$$0 \le \kappa_\phi := 2C(h)C_2(1 + C_1 M_\phi^{10}) < \nu,$$

then,

$$\|\mathscr{L}_t^{-1}\|_{(L^2(J;L^2(\Omega)), L^2(J;H_0^1(\Omega)))} \le \frac{\nu M_\phi^{10} + 2C(h) + 2C(h)C_1 M_\phi^{10}}{\nu - \kappa_\phi}, \tag{18}$$

where $C_1 := C_b + C_p\|c\|_{L^\infty(J;L^\infty(\Omega))}$, $C_2 := C_b + 4C(h)\|c\|_{L^\infty(J;L^\infty(\Omega))}$, $C_b := \|\sqrt{b_1^2 + \cdots + b_d^2}\|_{L^\infty(J;L^\infty(\Omega))}$.

In order to get an upper bound of M_ϕ^{10}, we need a priori bounds of a solution for the linear ODEs. Let $S^k(J) \subset V^1(J)$ be a finite element subspace on J with mesh size k. Then n-dimensional V^1-projection $P_k^1 : V^1(J)^n \to S^k(J)^n$ is defined for each $\mathrm{u} \in V^1(J)^n$ by

$$(\mathrm{u}, \mathrm{v}_k)_{V^1(J)^n} = (P_k^1\mathrm{u}, \mathrm{v}_k)_{V^1(J)^n}, \quad {}^\forall \mathrm{v}_k \in S^k(J)^n.$$

We also assume that
Assumption 2. There exists a constant $C_J(k)$ s.t.

$$\begin{aligned} \|\mathrm{u} - P_k^1\mathrm{u}\|_{V^1(J)^n} &\le C_J(k)\|\mathrm{u}''\|_{L^2(J)^n}, \quad {}^\forall \mathrm{u} \in V^1(J)^n \cap H^2(J)^n, \\ \|\mathrm{u} - P_k^1\mathrm{u}\|_{L^2(J)^n} &\le C_J(k)\|\mathrm{u} - P_k^1\mathrm{u}\|_{V^1(J)^n}, \quad {}^\forall \mathrm{u} \in V^1(J)^n. \end{aligned}$$

We denote the basis of $S^k(J)^n$ by $\{\psi_i\}_{i=1,\cdots,m}$. Let M_ψ^{01} be a positive constant such that

$$M_\psi^{01} := \left\| L_\psi^{T/2} G_\psi^{-1} D_\psi^{1/2} \right\|_2,$$

where, L_ψ, D_ψ, G_ψ are $m \times m$ matrices such that

$$L_{\psi ij} := (\psi_j, \psi_i)_{L^2(J)^n}, \quad D_{\psi ij} := (\psi_j', \psi_i')_{L^2(J)^n},$$

$$G_{\psi ij} := (L_\phi \psi_j', \psi_i')_{L^2(J)^n} + (Q_\phi \psi_j, \psi_i')_{L^2(J)^n}.$$

Noting that L_ϕ is symmetric and positive definite, let $A_c > 0$ be the minimum eigenvalue of L_ϕ. Then, by using the arguments in [7] with approximation property in Assumption 2, we have the following estimates corresponding to the semidiscrete inverse operator.

Theorem 3.2. *If*

$$0 \le \kappa_\psi := C_J(k)\|Q_\phi\|_{L^\infty(J)^{n,n}}(1 + M_\psi^{01}\|Q_\phi\|_{L^\infty(J)^{n,n}}) < A_c,$$

then it holds that

$$M_\phi^{10} \le \hat{M}_\psi^{01} + C_J(k)\frac{\left(1 + M_\psi^{01}\|Q_\phi\|_{L^\infty(J)^{n,n}}\right)^2}{Ac - \kappa_\psi}\|D_\phi^{T/2}\|_2\|L_\phi^{1/2}\|_2,$$

where $\hat{M}_\psi^{01} := \|L_\psi^{T/2}\hat{G}_\psi^{-1}D_\psi^{1/2}\|_2$, $\hat{G}_{\psi ij} := (L_\phi^{T/2}D_\phi^{-T/2}\psi_j', \psi_i')_{L^2(J)^n} + (L_\phi^{-1/2}Q_\phi D_\phi^{-T/2}\psi_j, \psi_i')_{L^2(J)^n}$ *and* $\|\cdot\|_2$ *means the matrix 2-norm, i.e., spectral norm.*

Note that $\hat{M}_\psi^{01}$ means an approximate norm of the inverse operator $\mathscr{L}_t^{-1}$.

3.3 A posteriori estimates II

We now introduce an alternative approach to the method in the previous subsection.

Constructive a priori error estimates for the heat equation In this paragraph, according to the arguments in [24], we give the constructive a priori error estimates for a full discrete numerical solution of the heat equation, which plays an esseential role in this subsection. This numerical scheme consisits of a Galerkin finite element method with interpolation in time using fundamental solutions for ODEs by spatial discretization. The present estimates is a basis for the verified computation of the inverse for the linearlized parabolic operators.

Let $S_h(\Omega) \subset H_0^1(\Omega)$ be a piecewise linear finite element subspace for spatial direction with $\dim S_h = n$. Also let $S^k(J) \subset V^1(J)$ be a piecewise linear finite element subspace for time direction with $\dim S^k = m$.

We define the interpolation $I^k : V^1(J; S_h(\Omega)) \to S^k(J; S_h(\Omega)) \equiv S_h(\Omega) \otimes S^k(J)$ by

$$u(t_i) = I^k(u(t_i)), \quad i = 1, \cdots, m.$$

Then we define the full-discrete projection $P_h^k : V \to S^k(J; S_h(\Omega))$ by

$$P_h^k u := I^k(P_h^V u). \tag{19}$$

For a given $f \in L^2(J; L^2(\Omega))$ consider the following equation with homogeneous initial and boundary conditions:

$$\tfrac{\partial}{\partial t}u - \nu\Delta u = f. \tag{20}$$

For a solution $u \in V$ of (20), $P_h^V u(x,t) \in V^1(J; S_h(\Omega))$ can be represented, by using a vector-valued function $\vec{u}_h \in V^1(J)^n$, as

$$P_h^V u(x,t) = \vec{u}_h(t)^T\Phi(x).$$

Here, $\Phi(x) \equiv (\phi_1, \cdots, \phi_n)^T$. The above equality is equivalent to

$$\begin{cases} L_\phi \dfrac{d}{dt}\vec{u}_h + \nu D_\phi \vec{u}_h = \tilde{f} & \text{in } J, \qquad (21a) \\ \vec{u}_h(0) = 0, & \qquad (21b) \end{cases}$$

where L_ϕ, D_ϕ are $n \times n$ matrices same as in the previous subsection and n-dimensional vector $\tilde{f} \equiv (\tilde{f}_i)$ is defined by $\tilde{f}_i = (f, \phi_i)_{L^2(\Omega)}$.

Noting that, by using the fundamental matrix for the ODE system, the solution of (21) can be rewitten as

$$\vec{u}_h(t) = \int_0^t \exp(\nu L_\phi^{-1} D_\phi (s-t)) L_\phi^{-1} \tilde{f}(s)\, ds, \tag{22}$$

which implies

$$(P_h^k u)(x,t_j) = \left(\int_0^{t_j} \exp(\nu L_\phi^{-1} D_\phi (s-t_j)) L_\phi^{-1} \tilde{f}(s)\, ds \right) \Phi(x), \quad \forall x \in \Omega, 1 \le j \le m. \tag{23}$$

Remark 3.3. *Let Ω be a rectangular domain, and take $S_h(\Omega)$ as a Q1 finite element space, namely, piecewize bi-linear element which is constituted by the tensor product of one dimensional P1 element. If we use the uniform mesh, then $L_\phi^{-1} D_\phi$ is a symmetric positive definite matrix(see [24]). Therefore, the diagonalization of $L_\phi^{-1} D_\phi$ is easily obtained in this case. In other cases, it is also diagonlizable([24]). Therefore, the computation of the exponential matrix functions in* (23) *is not so difficult.*

In what follows, for simplicity, we sometimes denote the symbols $L^2(J;H_0^1(\Omega))$ and $L^2(J;L^2(\Omega))$ by $L^2H_0^1$ and L^2L^2, respectively, and so on, which will cause no confusion.

Now let u be a solution of (20). Then observe that

$$\|u - P_h^k u\|_{L^2H_0^1} \le \|u - P_h^V u\|_{L^2H_0^1} + \|P_h^V u - P_h^k u\|_{L^2H_0^1}. \tag{24}$$

By existing a priori estimates for the semidisctre approximation([15]), we have

$$\|u - P_h^V u\|_{L^2H_0^1} \le \frac{2C(h)}{\nu} \|f\|_{L^2L^2}, \tag{25}$$

where $C(h)$ is the same constant in the subsection 3.2.
By the use of well-known inverse inequality for S_h, there exists a positive constant $C_{\mathrm{inv}}(h)$ satisfying

$$\left\| \nabla (P_h^V u - P_h^k u) \right\|_{L^2L^2} \le C_{\mathrm{inv}}(h) \left\| P_h^V u - P_h^k u \right\|_{L^2L^2}. \tag{26}$$

Note that, usually, $C_{\mathrm{inv}}(h) \approx O(h^{-1})$.

On the other hand, taking notice that the V^1-projection defined in the previous subsection coincides with the interpolation I^k on J, by using Assumption 2 and a priori estimates for the solution of (20), we have

$$\left\| P_h^V u - P_h^k u \right\|_{L^2L^2} \le C_J(k) \|f\|_{L^2L^2}. \tag{27}$$

Therefore,

$$\left\| P_h^V u - P_h^k u \right\|_{L^2H_0^1} \le C_{\mathrm{inv}}(h) C_J(k) \|f\|_{L^2L^2}. \tag{28}$$

Thus by (24),(25),(28), we obtain the following constructive error estimates for the full-discrete approximation to the solution of (20)

$$\|u - P_h^k u\|_{L^2H_0^1} \le C_1(h,k) \|f\|_{L^2L^2}, \tag{29}$$

where $C_1(h,k) := \dfrac{2C(h)}{\nu} + C_{\mathrm{inv}}(h) C_J(k)$.
If we take $k = h^2$, then (29) means an $O(h)$ estimate.

We also derive an L^2–estimate as below. Namely, first, by using the existing L^2-estimates for semidiscrete approximation ([23]), we have

$$\|u - P_h^V u\|_{L^2L^2} \leq \frac{8C(h)^2}{\nu} \|f\|_{L^2L^2}, \quad \forall u \in V. \tag{30}$$

On the other hand, we have by Assumption 2

$$\left\|P_h^V u - P_h^k u\right\|_{L^2L^2} \leq C_J(k) \|P_h^V u\|_{V^1\left(J;L^2(\Omega)\right)}, \tag{31}$$

which implies, by using the estimates $\|P_h^V u\|_{V^1\left(J;L^2(\Omega)\right)} \leq \|f\|_{L^2L^2}$,

$$\left\|P_h^V u - P_h^k u\right\|_{L^2L^2} \leq C_J(k) \|f\|_{L^2L^2}. \tag{32}$$

Thus, setting

$$C_0(h,k) := \frac{8C(h)^2}{\nu} + C_J(k),$$

we have

$$\|u - P_h^k u\|_{L^2L^2} \leq C_0(h,k) \|f\|_{L^2L^2}. \tag{33}$$

If $k = h^2$, then (33) gives the $O(h^2)$ estimates.

Estimation of the inverse operator In the present paragraph, we give an outline of the arguments in [8]. Combining the results obtained in the previous paragraph with an argument for the discretized inverse operator and a contraction property of the Newton-type formulation, we derive an a posteriori estimate for the operator $\mathscr{L}_t^{-1}$.

First, we set $\Delta_t := \frac{\partial}{\partial t} - \nu\Delta$ and $\mathscr{A} := -\Delta_t{}^{-1}(b\cdot\nabla) - \Delta_t{}^{-1}c$, where ν, b, c are same as before, and $\Delta_t{}^{-1}$ stands for the solution operator of the heat equation with homogeneous initial and boundary conditions. Next, we rewrite (15) as the following fixed point form

$$\begin{aligned} & \frac{\partial w}{\partial t} - \nu\Delta w + (b\cdot\nabla)w + cw = q \\ \Longleftrightarrow \qquad & w = \Delta_t^{-1}\left(q - (b\cdot\nabla)w - cw\right). \end{aligned} \tag{34}$$

Furthermore, we decompose it into the following two parts similar to the elliptic case in Section 2:

$$\left\{ \begin{array}{ll} P_h^k w = P_h^k\left(\Delta_t^{-1}q + \mathscr{A}w\right), & \text{(35a)} \\ (\mathscr{I} - P_h^k)w = (\mathscr{I} - P_h^k)\left(\Delta_t^{-1}q + \mathscr{A}w\right). & \text{(35b)} \end{array} \right.$$

Here, $\mathscr{I}$ means the identity map on $L^2(J;H_0^1(\Omega))$.
Note that (35a) can be rewritten as:

$$P_h^k w - P_h^k \mathscr{A} P_h^k w = P_h^k\left(\Delta_t^{-1}q + \mathscr{A}(\mathscr{I} - P_h^k)w\right).$$

Let define $M_{\phi,\psi}$ by

$$M_{\phi,\psi} := \|[\mathscr{I} - \mathscr{A}]_{hk}^{-1}\|_{\mathscr{L}(L^2(J;H_0^1(\Omega)),L^2(J;H_0^1(\Omega)))},$$

which can be computed by the matrix norm estimation corresponding to the finite dimensional operator $[\mathscr{I} - \mathscr{A}]_{hk}^{-1}$ on $S_h^k \equiv S_h(\Omega)\otimes S^k(J)$. Here, $[\mathscr{I} - \mathscr{A}]_{hk}^{-1}$ means the invese of $P_h^k(\mathscr{I} - \mathscr{A})|_{S_h^k}$ on S_h^k.

For detailed computational procedures of $M_{\phi,\psi}$, see [8].
Then, on the estimates for the finite dimensional part, we have

$$\|P_h^k w\|_{L^2H_0^1} \le M_{\phi,\psi}\|P_h^k(\Delta_t^{-1}q + \mathscr{A}(\mathscr{I} - P_h^k)w)\|_{L^2H_0^1},$$

which yields the estimates

$$\left\|P_h^k w\right\|_{L^2H_0^1} \le \mathscr{C}_2 \|b\|_{L^\infty L^\infty} \|w_\perp\|_{L^2H_0^1} + \mathscr{C}_2 \|c\|_{L^\infty L^\infty} \|w_\perp\|_{L^2L^2} + \mathscr{C}_2 \|q\|_{L^2L^2} \tag{36}$$

where we have set $w_\perp := (\mathscr{I} - P_h^k)w$ and $\mathscr{C}_2 := M_{\phi,\psi}\left(\frac{C_p}{\nu} + C_{\text{inv}}(h)C_J(k)\right)$.
On the other hand, for the infinite dimensional part, by the arguments using the a priori error estimates (29) and (33), we obtain the following inequality

$$\begin{aligned}\|w_\perp\|_{L^2H_0^1} &\le C_1(h,k)\mathscr{C}_3\left\|P_h^k w\right\|_{L^2H_0^1} + C_1(h,k)\|b\|_{L^\infty L^\infty}\|w_\perp\|_{L^2H_0^1} + C_1(h,k)\|q\|_{L^2L^2}\\ &+ C_1(h,k)\frac{C_0(h,k)\|c\|_{L^\infty L^\infty}}{1 - C_0(h,k)\|c\|_{L^\infty L^\infty}}\left(\mathscr{C}_3\left\|P_h^k w\right\|_{L^2H_0^1} + \|b\|_{L^\infty L^\infty}\|w_\perp\|_{L^2H_0^1} + \|q\|_{L^2L^2}\right),\end{aligned} \tag{37}$$

where $\mathscr{C}_3 \equiv C_1$, same constant as in Theorem 3.1.
Combining this estimates with (36), we get a system of two dimensional inequality with respect to $\|P_h^k w\|_{L^2H_0^1}$ and $\|w_\perp\|_{L^2H_0^1}$. Thus by solving it we obtain the following theorem.

Theorem 3.4. *Define the constant $\kappa_{\phi,\psi}$ by*

$$\kappa_{\phi,\psi} := \frac{\|b\|_{L^\infty L^\infty}(1 + \mathscr{C}_2\mathscr{C}_3)C_1(h,k) + \mathscr{C}_2\mathscr{C}_3C_0(h,k)\|c\|_{L^\infty L^\infty}}{1 - C_0(h,k)\|c\|_{L^\infty L^\infty}}. \tag{38}$$

If $0 \le \kappa_{\phi,\psi} < 1$, *then*

$$\|\mathscr{L}_t^{-1}\|_{\mathscr{L}(L^2(J;L^2(\Omega)),L^2(J;H_0^1(\Omega)))} \le \frac{1}{1 - \kappa_{\phi,\psi}}\frac{\mathscr{C}_2 + (1 + \mathscr{C}_2\mathscr{C}_3)C_1(h,k)}{1 - C_0(h,k)\|c\|_{L^\infty L^\infty}}. \tag{39}$$

3.4 Numerical Comparisons

We show some numerical results by three kinds of methods, namely, a priori estimates (16), a posteriori estimates I (18) and II (39). Also other comparison results are presented in [8].

We considered the norm estimates for an inverse operator of the following $\mathscr{L}_t$ with $\Omega = (0,1)$ and $J = (0,T)$, i.e., one space dimensional case,

$$\mathscr{L}_t := \frac{\partial}{\partial t} - \nu\Delta - 2u_h^k, \tag{40}$$

which implies that $b = 0$ and $c = -2u_h^k$ in (14). Here, the function $u_h^k \in V^1(J;L^2(\Omega)) \cap L^2(J;H_0^1(\Omega) \cap H^2(\Omega))$ is chosen as an approximation of the function $u(x,t) \equiv 0.5t\sin(\pi x)$ by using a piecewise-cubic and piecewise-linear interpolations in space and time, respectively. And the constant ν is set as $\nu = 0.1$ (Example 1) and $\nu = 1.0$ (Example 2).
In the application of the a posteriori estimates, we used the finite-dimensional spaces $S_h(\Omega)$ and $S^k(J)$, spanned by piecewise linear functions with uniform mesh size h and k, respectively. Then, it is seen that the constants in previous subsections could be taken as $C(h) = h/\pi$, $C_{\text{inv}}(h) = \sqrt{12}/h$, $C_J(k) = k/\pi$, and $C_p = 1/\pi$.

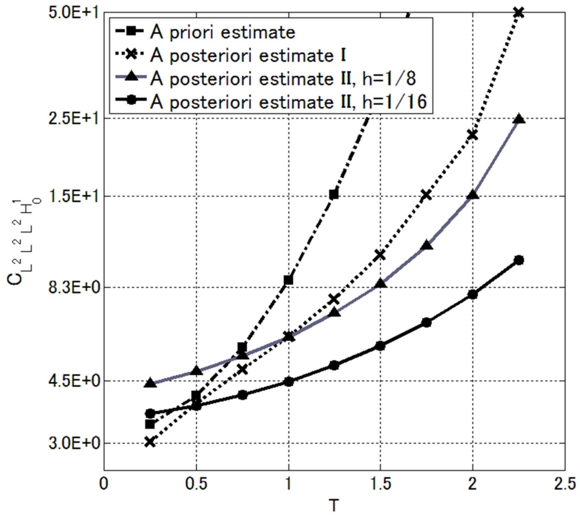

Figure 1: $\nu = 0.1$

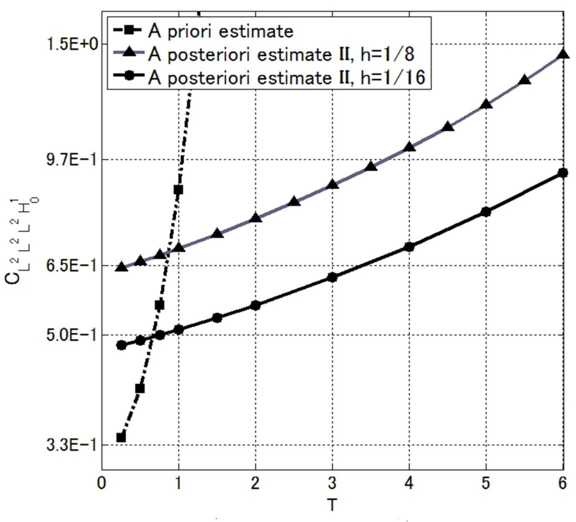

Figure 2: $\nu = 1$

For the a posteriori estimates I, i.e., (18), the finite element meshes were taken as $h = \frac{1}{6}, k = \frac{1}{700 \cdot T^2}$ for Example 1. On the other hand, in the a posteriori estimates II, i.e., (39), we chose meshes $h = 1/8$ and $h = 1/16$, with $k = h^2$.

Figures 1 and 2 show the estimated norms $\|\mathscr{L}_t^{-1}\|_{\mathscr{L}\left(L^2(J;L^2(\Omega)),L^2(J;H_0^1(\Omega))\right)}$ for Example 1 and 2, plotted out on log-linear coordinates.

In case of $\nu = 1$, i.e., Example 2, due to the stiffness of the corresponding ODEs in the estimation process, we were not successful in computing the inverse operator by a posteriori estimates I, except that T was very small.

From the above computational results, we could conclude:

- A posteriori methods give more accurate value than existing a priori estimate. Particularly, there is a possibility to remove the exponential dependency on time, even if the corresponding elliptic problem is not coercive.

- As far as the test problems are concerned, the a posteriori estimates II (39) seems to give finer bounds and to be more efficient in computational cost than the a posteriori estimates I (18) in many cases.

3.5 Verification of solutions for nonlinear parabolic problrems

In the present subsection, as an application of the a posteriori estimates II, i.e., the inequality (39), we describe on the verified computations of solutions for nonlinear parabolic equations with some numerical examples. In order to show more clearly the detailed verification procedures, we consider the following prototype problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu\Delta u = u^2 + f, & \text{in } \Omega \times J, \quad (41a) \\ u(x,t) = 0, & \text{on } \partial\Omega \times J, \quad (41b) \\ u(x,0) = 0, & \text{in } \Omega. \quad (41c) \end{cases}$$

In the below, Ω and J are the same as in the previous subsection. And the function f is chosen so that the problem (41) has the exact solution $u(x,t) = 0.5t\sin(\pi x)$, and the constant ν as $\nu = 0.1$ and

$\nu = 1.0$ which are corresponding to Example 1($\nu = 0.1$) and Example 2($\nu = 1$), respectively, in 3.4. Also u_h^k is taken as the same interpolation for u as before.

First, we consider the following residual equation:

$$\begin{cases} \mathscr{L}_t w \equiv \dfrac{\partial w}{\partial t} - \nu \Delta w - 2u_h^k w = g(w), & \text{in } \Omega \times J, \quad (42a) \\ w(x,t) = 0, & \text{on } \partial\Omega \times J, \quad (42b) \\ w(x,0) = 0, & \text{in } \Omega, \quad (42c) \end{cases}$$

where

$$g(w) := w^2 + \varepsilon(x,t), \quad \varepsilon(x,t) := (u_h^k)^2 + f - \left(\frac{\partial u_h^k}{\partial t} - \nu \Delta u_h^k\right).$$

Note that if the approximate solution u_h^k well approximates the exact solution of (41), then $w \approx 0$, $\varepsilon \approx 0$, and $g(w) \approx 0$. Thus, as described in the beginning of Section 3, we have the following fixed-point equation of the compact map F on $L^2(J; H_0^1(\Omega))$:

$$w = \mathscr{L}_t^{-1} g(w) =: F(w). \tag{43}$$

In the actual application of the verification principle, we need the norm estimates in the space $V^1(J; L^2)$. Therefore, for any positive constants α and β, we define the candidate set $W_{\alpha,\beta}$ as

$$W_{\alpha,\beta} := \left\{ w \in V \ ; \ \|w\|_{L^2 H_0^1} \leq \alpha, \ \|w\|_{V^1 L^2} \leq \beta \right\}, \tag{44}$$

which is a bit of different from the set defined in (11). Taking notice of the continuity of the map F in the space $L^2(J; H_0^1(\Omega))$, by Schauder's fixed-point theorem, if $W_{\alpha,\beta}$ satisfies

$$F(W_{\alpha,\beta}) \subset W_{\alpha,\beta}, \tag{45}$$

then a fixed point of (43) exists in the set $\overline{W_{\alpha,\beta}}$, where $\overline{W_{\alpha,\beta}}$ stands for the closure of the set $W_{\alpha,\beta}$ in $L^2(J; H_0^1(\Omega))$.

Now, setting $C_{L^2L^2, L^2H_0^1} := \|\mathscr{L}_t^{-1}\|_{\mathscr{L}(L^2L^2, L^2H_0^1)}$, by some simple calculations using the Sobolev embedding theorem and the Poincaré inequality, it is easily seen that the following inequalities hold for any $w \in W_{\alpha,\beta}$:

$$\|F(w)\|_{L^2H_0^1} \leq C_{L^2L^2, L^2H_0^1} \left(\alpha\beta\sqrt{\tfrac{T}{8}} + \|\varepsilon\|_{L^2L^2} \right),$$

$$\|F(w)\|_{V^1L^2} \leq \left(\frac{2}{\pi} C_{L^2L^2, L^2H_0^1} \left\|u_h^k\right\|_{L^\infty L^\infty} + 1 \right) \left(\alpha\beta\sqrt{\tfrac{T}{8}} + \|\varepsilon\|_{L^2L^2} \right).$$

From these inequalities, we have the following sufficient condition for (45):

$$\begin{cases} C_{L^2L^2, L^2H_0^1} \left(\alpha\beta\sqrt{\tfrac{T}{8}} + \|\varepsilon\|_{L^2L^2} \right) \leq \alpha, \\ \left(\dfrac{2}{\pi} C_{L^2L^2, L^2H_0^1} \left\|u_h^k\right\|_{L^\infty L^\infty} + 1 \right) \left(\alpha\beta\sqrt{\tfrac{T}{8}} + \|\varepsilon\|_{L^2L^2} \right) \leq \beta. \end{cases}$$

Solving the above simultaneous algebraic inequalities in α and β, we have the error bounds of the form $\|u - u_h^k\|_{L^2H_0^1} \leq \alpha$ and $\|u - u_h^k\|_{V^1L^2} \leq \beta$. We show the verification results for the solutions of (42) in Table I. In the table, 'Residual'$\equiv \|\varepsilon\|_{L^2L^2}$ and $C_{\mathscr{L}_t^{-1}} \equiv C_{L^2L^2, L^2H_0^1}$.

From this table, it is seen that the error bounds increase in proportion to the residual norms. This property should be expected in our verification conditions. Namely, the validated accuracy of the present method is essentially dependent on the residual norm of the approximate solutions.

Table 1: $u(x,t) = 0.5\, t\, \sin(\pi x)$, $h = 1/8$, $k = h^2$.

ν	T	Residual	$C_{\mathscr{L}_t^{-1}}$	α	β
0.1	0.25	0.0001	4.4030	0.0006	0.0002
	0.50	0.0004	4.7701	0.0020	0.0010
	1.00	0.0012	5.9881	0.0071	0.0060
	1.50	0.0021	8.4710	0.0255	0.0446
1.0	0.25	0.0014	0.6450	0.0009	0.0016
	0.50	0.0041	0.6595	0.0027	0.0050
	1.00	0.0116	0.6932	0.0081	0.0170
	2.00	0.0327	0.7752	0.0268	0.0764
	2.50	0.0457	0.8240	0.0431	0.1542

3.6 A possible refinement of inverse norm estimates

In this subsection, we consider some refinement on the inverse norm estimates in Theorem 3.4. Note that if we take the mesh size such as $k \approx h$, then the constant $C_1(h,k)$ in (38) implies $C_1(h,k) \approx O(1)$ by the definition (29). Hence in case that $b \neq 0$, the numerator in the right-hand side of (38) might be not so small even if we use fine mesh, i.e., h is sufficiently small. Therefore, in such a case it should be difficult to attain the condition $\kappa_{\phi,\psi} < 1$ in Theorem 3.4. On the other hand, as shown in (33), the L^2 error estimates of the projection P_h^k still remains $O(h^2) + O(k)$ independent of the relation between h and k.

In the below, we consider a method to obtain the invertibility condition which is always attained for sufficiently small h and k even if $k \approx h$.

Now we consider again the linear parabolic equation: $\Delta_t u = f - (b \cdot \nabla)u - cu$ which is corresponding equation to (34) and the following decomposed form as in (35a), (35b).

$$\begin{cases} P_h^k u = P_h^k(\Delta_t^{-1} f + \mathscr{A} u), & (47\text{a}) \\ (\mathscr{I} - P_h^k)u = (\mathscr{I} - P_h^k)(\Delta_t^{-1} f + \mathscr{A} u). & (47\text{b}) \end{cases}$$

From (47a), using the definition of the operator $\mathscr{A}$, we have

$$P_h^k u = P_h^k\left(\mathscr{A}(P_h^k u + u_\perp) + \Delta_t^{-1} f\right).$$

By the definition of the operator $[\mathscr{I} - \mathscr{A}]_{hk}^{-1}$, this implies

$$P_h^k u = [\mathscr{I} - \mathscr{A}]_{hk}^{-1} P_h^k\left(\mathscr{A} u_\perp + \Delta_t^{-1} f\right). \tag{48}$$

Now we set as

$$M^{0,0} := \left\| [\mathscr{I} - \mathscr{A}]_{hk}^{-1} \right\|_{\mathscr{L}(L^2(J;L^2(\Omega)),L^2(J;L^2(\Omega)))}, \tag{49}$$

which can be estimated by the following matrix norm

$$M^{0,0} = \left\| L_{\phi,\psi}^{T/2} G_{\phi,\psi}^{-1} L_{\phi,\psi}^{-T/2} \right\|_2.$$

Thus we have by (48)

$$\left\|P_h^k u\right\|_{L^2L^2} \le M^{0,0}\left(\left\|P_h^k \mathscr{A} u_\perp\right\|_{L^2L^2} + \left\|P_h^k \Delta_t^{-1} f\right\|_{L^2L^2}\right). \tag{50}$$

We now estimate the first term in the right-hand side of (50).
First, we set $\psi := \mathscr{A} u_\perp$, then noting that $\Delta_t \psi = -b\cdot\nabla u_\perp - cu_\perp$ we have

$$\begin{aligned}\int_0^T \{(\psi_t,\psi)+\nu(\nabla\psi,\nabla\psi)\}dt &= \int_0^T (-b\cdot\nabla u_\perp - cu_\perp,\psi)dt\\ &= \int_0^T (u_\perp,\nabla\cdot(b\psi)-c\psi)dt\\ &\le \|u_\perp\|\,(C_p|\nabla\cdot b|+\mathscr{C}_3)\,\|\nabla\psi\|.\end{aligned} \tag{51}$$

Here, and in what follows, we suppress the space dependency in the symbols for the L^2-norm, namely, $||\cdot||_{L^2L^2}$ is denoted as $||\cdot||$, and also for a function $\phi\in L^\infty(\Omega\times J)$, we simply denote $|\phi|\equiv||\phi||_{L^\infty(\Omega\times J)}$.
From (51), taking notice of the initial condition in the integration of the first term of left-hand side, we obtain

$$\begin{aligned}\|\nabla\psi\| &\le \frac{1}{\nu}(C_p|\nabla\cdot b|+\mathscr{C}_3)\,\|u_\perp\|\\ &= K\,\|u_\perp\|,\end{aligned} \tag{52}$$

where $K := \frac{1}{\nu}(C_p|\nabla\cdot b|+\mathscr{C}_3)$.
We also have the following lemma.

Lemma 3.5. *For a solution u of $\Delta_t u = f-(b\cdot\nabla)u-cu$, it holds that*

$$||\nabla u|| \le \frac{\mathscr{C}_3}{\nu}||u|| + \frac{C_p}{\nu}||f||. \tag{53}$$

Indeed, we have by the assumption

$$\int_0^T \{(u_t,u)+\nu(\nabla u,\nabla u)+(b\cdot\nabla u+cu,u)\}dt = \int_0^T (f,u)dt.$$

Therefore, taking account of the initial condition, it is easy to obtain the following inequality

$$\nu||\nabla u|| \le (|b|+C_p|c|)||u||+C_p||f||,$$

which proves the lemma.
Next, by using the estimates (33), (29) and the definition of ψ, we have

$$\begin{aligned}\left\|P_h^k\psi\right\| &\le \|\psi\| + C_0(h,k)\,\|\Delta_t\psi\|\\ &\le C_p\,\|\nabla\psi\| + C_0(h,k)\,\|-b\cdot\nabla u_\perp - cu_\perp\|\\ &\le C_p\,\|\nabla\psi\| + C_0(h,k)\mathscr{C}_3\,\|\nabla u_\perp\|\\ &\le C_p\,\|\nabla\psi\| + C_0(h,k)\mathscr{C}_3 C_1(h,k)\,\|\Delta_t u\|\\ &= C_p\,\|\nabla\psi\| + C_0(h,k)\mathscr{C}_3 C_1(h,k)\,\|-b\cdot\nabla u - cu + f\|.\end{aligned} \tag{54}$$

Here, $C_0(h,k)$ and $C_1(h,k)$ are same constants in 3.3.
Furthermore, by using Lemma 3.5, we get

$$\begin{aligned}\|-b\cdot\nabla u - cu + f\| &\le |b|\,\|\nabla u\| + |c|\,\|u\| + \|f\|\\ &\le |b|(\frac{\mathscr{C}_3}{\nu}||u|| + \frac{C_p}{\nu}||f||) + |c|||u|| + ||f||\\ &\le (|b|\frac{\mathscr{C}_3}{\nu}+|c|)||P_h^k u|| + (|b|\frac{\mathscr{C}_3}{\nu}+|c|)||u_\perp|| + (|b|\frac{C_p}{\nu}+1)||f||.\end{aligned}$$

Therefore, from (52) and (54), we have

$$\left\|P_h^k\psi\right\| \le C_pK\|u_\perp\| + C_0(h,k)\mathscr{C}_3C_1(h,k)\{(|b|\frac{\mathscr{C}_3}{\nu}+|c|)\|P_h^ku\| + (|b|\frac{\mathscr{C}_3}{\nu}+|c|)\|u_\perp\| + (|b|\frac{C_p}{\nu}+1)\|f\|\}. \tag{55}$$

We now set

$$C^{(0)}(h,k) := C_0(h,k)\mathscr{C}_3C_1(h,k)(|b|\frac{\mathscr{C}_3}{\nu}+|c|),$$
$$C^{(2)}(h,k) := C_0(h,k)\mathscr{C}_3C_1(h,k)(|b|\frac{C_p}{\nu}+1).$$

Then, we have

$$\left\|P_h^k\mathscr{A}u_\perp\right\| \le C^{(0)}(h,k)\{\|P_h^ku\| + (C_pK + C^{(0)}(h,k))\|u_\perp\| + C^{(2)}(h,k)\|f\|\}. \tag{56}$$

We also have the following estimate by Lemma 5.3 in [24]

$$\begin{aligned}\left\|P_h^k\Delta_t^{-1}f\right\|_{L^2L^2} &\le C_p\left\|P_h^k\Delta_t^{-1}f\right\|_{L^2H_0^1}\\ &\le C_p(\frac{C_p}{\nu}+C_{\mathrm{inv}}(h)C_J(k))\|f\|.\end{aligned}$$

Hence by (50) and (56) imply the following estimate as the finite dimensional part

$$\begin{aligned}\left\|P_h^ku\right\| &\le M^{0,0}\{C^{(0)}(h,k)\|P_h^ku\| + (C_pK+C^{(0)}(h,k))\|u_\perp\| + (C_p(\frac{C_p}{\nu}+C_{\mathrm{inv}}(h)C_J(k)) + C^{(2)}(h,k))\|f\|\}\\ &= O_{11}\|P_h^ku\| + (Q_1+O_{12})\|u_\perp\| + (Q_2+O_{13})\|f\|,\end{aligned} \tag{57}$$

where we have defined the constants $O_{11} \equiv O_{12} := M^{0,0}C^{(0)}(h,k)$, $O_{13} := M^{0,0}C^{(2)}(h,k)$ and $Q_1 := M^{0,0}C_pK$, $Q_2 := M^{0,0}C_p(\frac{C_p}{\nu}+C_{\mathrm{inv}}(h)C_J(k))$.
On the other hand, by considering the L^2L^2 norm of (47b), from Theorem 3.2 in [8] and Lemma 3.5, we have

$$\begin{aligned}\|u_\perp\|_{L^2L^2} &\le C_0(h,k)\,\|-(b\cdot\nabla)u - cu + f\|_{L^2L^2}\\ &\le C_0(h,k)\,(|b|\,\|\nabla u\| + |c|\,\|u\| + \|f\|)\\ &\le C_0(h,k)\left(|b|(\frac{\mathscr{C}_3}{\nu}\|u\| + \frac{C_p}{\nu}\|f\|) + |c|\,\|u\| + \|f\|\right)\\ &\le C_0(h,k)\left((\frac{|b|\mathscr{C}_3}{\nu}+|c|)\|u\| + (\frac{|b|C_p}{\nu}+1)\|f\|\right)\\ &\le O_{21}\left\|P_h^ku\right\| + O_{22}\,\|u_\perp\| + O_{23}\|f\|,\end{aligned} \tag{58}$$

where we have defined the constants $O_{21} \equiv O_{22} := C_0(h,k)(\frac{|b|\mathscr{C}_3}{\nu}+|c|)$ and $O_{23} := C_0(h,k)(\frac{|b|C_p}{\nu}+1)$.
Thus from (57) and (58), we get the following simultaneous inequality,

$$\begin{pmatrix}1-O_{11} & -Q_1-O_{12}\\ -O_{21} & 1-O_{22}\end{pmatrix}\begin{pmatrix}\|P_h^ku\|\\ \|u_\perp\|\end{pmatrix} \le \begin{pmatrix}Q_2+O_{13}\\ O_{23}\end{pmatrix}\|f\|. \tag{59}$$

Now, observe that

$$\det\begin{pmatrix}1-O_{11} & -Q_1-O_{12}\\ -O_{21} & 1-O_{22}\end{pmatrix} = 1 - O_{11} - O_{22} - Q_1O_{21}.$$

Therefore, defining

$$\kappa^0_{\phi,\psi} := O_{11} + (1+Q_1)O_{22}, \tag{60}$$

if $1-\kappa^0_{\phi,\psi} > 0$, then the above determinant is positive and the inequality (59) can be solved, which yields the following L^2-based inverse norm estimates.

Theorem 3.6. *Let O_{ij}, Q_i and $\kappa^0_{\phi,\psi}$ be constants defined above. If $\kappa^0_{\phi,\psi} < 1$, then the following estimates hold.*

$$\begin{pmatrix} \|P_h^k u\| \\ \|u_\perp\| \end{pmatrix} \le \frac{1}{1-\kappa^0_{\phi,\psi}} \begin{pmatrix} 1-O_{22} & Q_1+O_{11} \\ O_{22} & 1-O_{11} \end{pmatrix} \begin{pmatrix} Q_2+O_{13} \\ O_{23} \end{pmatrix} \|f\|. \tag{61}$$

This estimates will give the desired result. Namely, since the constants O_{11} and $O_{22} \to 0$ when $h \to 0$ even if we take $k = h$, it holds that $1-\kappa^0_{\phi,\psi} > 0$ for sufficiently small h. Therefore, we can effectively compute the L^2 norm of the inverse operator, while it might occur that Theorem 3.4 does not work in the case of $b \neq 0$. In that sense, we could say this result should be an essential improvement of the estimates in Theorem 3.4. Thus it is also expected that Theorem 3.6 presents a more efficient verification techniques for nonlinear problems.

4 Periodic problems

4.1 The problem with known period

As discussed above, in order to implement the infinite dimensional Newton-type method, it is essential to estimate the norm for the linearlized inverse opeartor. Therefore, we consider the following linear parabolic problem with time-periodic condition:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu\Delta u + (b\cdot\nabla)u + cu = f(x,t), & \text{in } \Omega\times J, \quad (62a) \\ u(x,t) = 0, & \text{on } \partial\Omega\times J, \quad (62b) \\ u(x,0) = u(x,T), & \text{in } \Omega, \quad (62c) \end{cases}$$

Therefore, it will be essential and important to get a constructive error estimates for a full-discrete approximation to the time-periodic solution of the following simple problem with any $f \in L^2(J;L^2(\Omega))$

$$\begin{cases} \frac{\partial}{\partial t}u - \nu\Delta u = f, & \text{in } \Omega\times J, \quad (63a) \\ u(x,t) = 0, & \text{on } \partial\Omega\times J, \quad (63b) \\ u(x,0) = u(x,T), & \text{in } \Omega. \quad (63c) \end{cases}$$

Note that the existence and uniqueness of a weak solution for the above problem is well known, e.g., [31]. Let $S_h(\Omega) \subset H^1_0(\Omega)$ be the same finite element subspace in the section 3.3. Also let $\tilde{S}^k(J) \subset \tilde{V}^1(J) := \{u \in V^1(J) \mid u(0) = u(T)\}$ be a piecewise linear finite element subspace for time direction.

Then a full-discrete approximation $P_h^k : V \to \tilde{S}^k(J;S_h(\Omega))$ of a solution u of the problem (63) can be defined same as in (19):

$$P_h^k u := I^k(P_h^V u).$$

Namely, we consider the following ODEs with constant coefficients which is from the semidiscrete approximation of (63)

$$\begin{cases} L_\phi \dfrac{d}{dt}\vec{u}_h + \nu D_\phi \vec{u}_h = \tilde{f} & \text{in } J, \quad (64a) \\ \vec{u}_h(0) = \vec{u}_h(T). & \quad (64b) \end{cases}$$

Here, $L_\phi, D_\phi \in R^{n\times n}$ and $\vec{f} \in R^n$ are same as defined in (21). Then by using the fundamental matrix: $\Theta(t) = \exp(-\nu L_\phi^{-1} D_\phi t)$ for the operator $L_\phi \frac{d}{dt} + \nu D_\phi$ with setting $\vec{b}(t) = L_\phi^{-1}\vec{f}(t)$, we have

$$\begin{cases} \vec{u}_h(t) = \Theta(t)\vec{u}_h(0) + \int_0^t \Theta(t-s)\vec{b}(s)\, ds \quad \text{in } J, & (65a) \\ \vec{u}_h(0) = \vec{u}_h(T). & (65b) \end{cases}$$

Assuming that $(I - \Theta(T))$ is nonsingular, by considering the periodic condition, we have by (65)

$$\vec{u}_h(t) = \Theta(t)\,(I-\Theta(T))^{-1} \int_0^T \Theta(T-s)\vec{b}(s)\, ds + \int_0^t \Theta(t-s)\vec{b}(s)\, ds. \tag{66}$$

Therefore,

$$P_h^k u(x,t_j) = \left(\Theta(t_j)\,(I-\Theta(T))^{-1} \int_0^T \Theta(T-s)\vec{b}(s)\, ds + \int_0^{t_j} \Theta(t_j-s)\vec{b}(s)\, ds\right)\Phi(x).$$

We can also derive the $L^2(J;H_0^1(\Omega))$ and $L^2(J;L^2(\Omega))$ error estimates for $P_h^k u$ as before to formulate the verification condition of nonlinear periodic problems.

4.2 The problem with unknown period

We consider a basic formulation of the verification of time-periodic solutions for parabolic problems with unknown period. Denote the nonlinear parabolic problem with unknown period T by

$$\begin{cases} u_t - \nu\Delta u &= f(x,t,u) \quad \text{in } \Omega\times(0,T), \\ u(x,0) &= u(x,T) \quad \text{in } \Omega. \end{cases} \tag{67}$$

Here, we try to find a solution such that $u(\cdot,t) \in H_0^1(\Omega)$ for any $t \in (0,T)$. Then, using the transformation $s := \frac{t}{T}$ and setting $\hat{u}(s) := u(Ts)$ we have for $J \equiv (0,1)$

$$\begin{cases} \frac{1}{T}\hat{u}_s - \nu\Delta\hat{u} &= f(x,Ts,\hat{u}(s)) \quad \text{in } \Omega\times J, \\ \hat{u}(x,0) &= \hat{u}(x,1) \quad \text{in } \Omega, \\ T &= T - \nu\|\nabla\hat{u}\|_{L^2(Q)} + \int_0^1 (f(x,Ts,\hat{u}(s)),\hat{u}(s))_\Omega ds. \end{cases} \tag{68}$$

Here, the third equation is followed by multiplying both side of the first equation by $\hat{u}$ and integrating on $Q := \Omega\times J$ taking account of the periodic condition in (68).
In order to formulate a Newton-type method, rewrite the above as follows:

$$\text{find } \exists\hat{u} \in H^1(J;L^2)\cap L^2(J:H_0^1)\cap\{\hat{u}(\cdot,0)=\hat{u}(\cdot,1)\},\ \exists T>0 \quad satisfying$$

$$\begin{cases} \hat{u}_s - T\nu\Delta\hat{u} - Tf(x,Ts,\hat{u}(s)) &= 0 \quad \text{in } Q, \\ \nu\|\nabla\hat{u}\|_Q^2 - \int_0^1 (f(x,Ts,\hat{u}(s)),\hat{u}(s))_\Omega ds &= 0. \end{cases} \tag{69}$$

By differentiating (69), we have the following linearized problem. Namely, for each $\phi \in L^2(J;L^2)$ and $r \in R^1$

$$\text{find } \exists v \in H^1(J;L^2)\cap L^2(J:H_0^1)\cap\{v(\cdot,0)=v(\cdot,1)\},\ \exists\tau>0 \quad satisfying$$

$$\begin{cases} v_s - \kappa_0 \Delta v + b \cdot \nabla v + cv + e\tau & = & \phi \quad \text{in} \quad Q, \\ (g_1, \nabla v)_Q + (g_0, v)_Q + \delta\tau & = & r. \end{cases} \tag{70}$$

Here, κ_0, δ are constants, and b, c, e, g_0, g_1 are scalar or vector functions in $(x,s) \in Q$ which are determined by an approximate solution of (69). Also $(\cdot,\cdot)_Q$ denotes the L^2 inner product on Q. The constructive a priori estimates for the solution (v,τ) of (70) will be possible by combining the arguments in the previous subsection for known period with the tecniques presented in [4]. Therefore, we can give a norm estimation for the linearized operator defined by the left-hand side of (70).

Thus, it will be possible to derive a Newton-type verification condition for the solution $\hat{u}$ and the unknown period T, in which the constructive error estimates for the projection P_h^k in the previous subsection will also play an essential role.

5 Conclusion

We considered some enclosure methods for solutions of parabolic initial-boundary value problems. Our method is based on the finite element approximation and the constructive error estimates for the simple heat equation. Particularly, the author believes that the full discrete method in the section 3.3 using the fundamental matrix obtained by spatial approximation should be sufficiently practical and useful from the viewpoint of computational efficiency. It implies that the verification principle for elliptic problems could also be used to enclose the solutions of evolutional equations. Moreover, we emphasize that our method has an advantage by the use of finite element method, because it enables us to apply the method to problems with arbitrary polygonal or polyhedral spatial domains. Therefore, it has more wider application fields than the methods based on the spectral method as in [1, 2, 32]. However, it is still in a germinal stage and it will be necessary to apply the technique to more realistic parabolic problems including the periodic solutions in the section 4 and to comfirm the efficiency of the verified computation.

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